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Static and dynamic fracture mechanics analysis of a DCB specimen considering shear deformation effects

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Abstract

Static and dynamic fracture is analyzed in a DCB specimen considering shear deformation effects. The DCB specimen is assumed to consist of two Timoshenko beams bonded together along a common edge except at the crack surfaces and is subjected to two splitting forces. Because of symmetry, only one half of the model is considered. So, the physical model of the problem consists of a Timoshenko beam lying on an elastic Winkler foundation. The problem is solved analytically and the stress intensity factor is derived in the static case, in general. However, in a special case when the uncracked ligament is large compared with the beam thickness, a simple closed form expression is derived for the stress intensity factor. The results are compared with those cited in the literature and a good agreement is observed. Finally, in the dynamic case, energy release rate and crack propagation velocity are derived and the effect of geometry and material property of the DCB is studied on the crack growth velocity.

Keywords: Fracture mechanics; DCB; Timoshenko beam; Stress intensity factor; Crack propagation velocity

1. Introduction

A test specimen for fracture mechanics investigation which is used extensively in both experimental and analytical points of view, is the double cantilever beam (DCB) specimen (Fig. 1).

So far, several expressions for the stress intensity factor have been presented in the literature. In the limiting case of $\frac{a}{h} \to \infty$, Gilman (1959) has found a simple relation for K by elementary beam theory

$$K = 2\sqrt{3}\frac{Pa}{bh^{3/2}} \tag{1}$$

In the other limiting case of $\frac{a}{h} \to 0$, K approaches Irwin's solution (1957) for the infinite sheet containing an edge crack loaded by tensile concentrated forces

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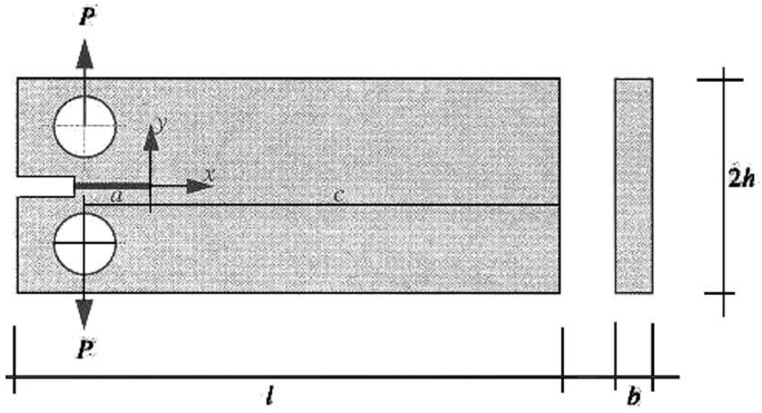

Fig. 1. Schematic view of a DCB specimen.

$$K = \frac{P}{b}\left(\frac{2}{\pi a}\right)^{1/2} \tag{2}$$

Gross and Srawley (1966) extended the elementary beam solution for smaller values of $\frac{a}{h}$ using a boundary collocation technique and obtained the following relation for K

$$K = 2\sqrt{3}\frac{Pa}{bh^{3/2}}\left(1 + 0.687\frac{h}{a}\right) \tag{3}$$

But the major development in determining K from beam theory models was performed by Kanninen (1973) by introducing a beam model which takes the region beyond the crack tip into account. In this model, as a result of symmetry, the upper half of specimen will be regarded as an Euler–Bernoulli beam which is supported by an elastic Winkler foundation (Kerr, 1964) with stiffness k at the ligament with length c, shown in Fig. 2. Finally, the stress intensity factor for constant force conditions is found to be

$$K = 2\sqrt{3}\frac{P}{\lambda bh^{3/2}}\left[\lambda a\left(\frac{\sinh^2 \lambda c + \sin^2 \lambda c}{\sinh^2 \lambda c - \sin^2 \lambda c}\right) + \left(\frac{\sinh \lambda c \cosh \lambda c - \sin \lambda c \cos \lambda c}{\sinh^2 \lambda c - \sin^2 \lambda c}\right)\right] \tag{4}$$

where $\lambda = \frac{6^{\frac{1}{4}}}{h}$.

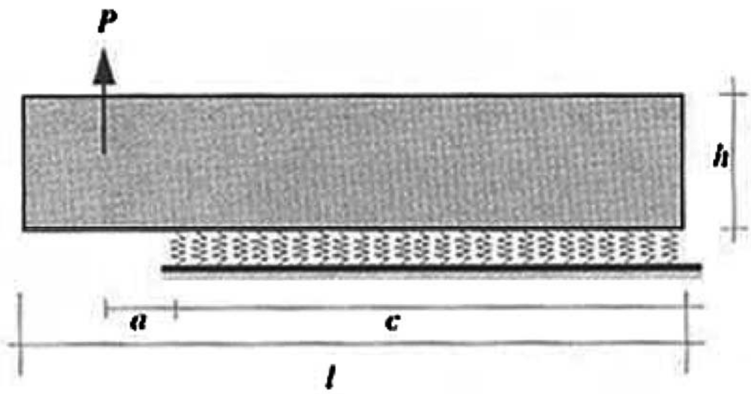

Fig. 2. Beam model of a DCB lying on Winkler support.

Kanninen used the following simple relation for the foundation stiffness k

$$k = \frac{2Eb}{h} \tag{5}$$

When the crack tip is far enough from the end of specimen ($c > 2h$), the solution for K (Eq. (4)) reduces to:

$$K = 2\sqrt{3}\frac{Pa}{bh^{3/2}}\left(1 + 0.64\frac{h}{a}\right) \tag{6}$$

Freiman et al. (1973) take a different method. In this method the beam is split into two parts: crack surface and the elastic foundation part. An energy approach is used to establish the stress intensity factor. For the first piece the energy due to shear stress is taken into account in addition to deformation due to bending of the beam. For the analysis of the second piece, the classical problem of a beam on an elastic foundation is used. The energy release rate for the whole specimen is determined by adding up the energy release rate of both parts. Finally K is found to be:

$$K = 2\sqrt{3}\frac{Pa}{bh^{3/2}}\left[\left(1 + 0.64\frac{h}{a}\right)^2 + \frac{1+v}{6}\left(\frac{h}{a}\right)^2\right]^{1/2} \tag{7}$$

where v is the Poisson's ratio. The second term in (7) is due to the shear stress of the beam.

Foote and Buchwald (1985) assumed the DCB specimen loaded by splitting forces, to be a special case of the arbitrarily loaded semi-infinite strip. This assumption is valid if the length of the uncracked ligament is greater than $2h$. The analysis is performed using the Wiener–Hopf technique, first used by Fichter (1983). Finally, a simple approximate formula for K could be found which differs from the theoretical K values by less than 1.1%:

$$K = 2\sqrt{3}\frac{Pa}{bh^{3/2}}\left(1 + 0.673\frac{h}{a}\right) + \frac{P}{b}\sqrt{\frac{2}{\pi a}} - \frac{P}{\sqrt{h}\left[0.815\left(\frac{a}{h}\right)^{0.619} + 0.429\right]} \tag{8}$$

Also, Kanninen (1974) has presented a formula for a DCB specimen, using a Timoshenko beam and a Pasternak foundation (Kerr, 1964). But, his work suffers the following disadvantages:

First, Gehlen et al. (1979) showed that the torsional stiffness of the foundation springs vanish in an "augmented beam model". Therefore, Winkler foundation is a better way for modeling. Second, the results are valid only when $c > 2h$.

In the present work, the problem of determining the stress intensity factor considering shear deformation, is investigated. A model which includes shear deformation effects is the Timoshenko beam theory. In the determination of stress intensity factor, no assumption is made about the uncracked length c, which is very important especially for studying specimens with deep flaws. However, a closed form solution is given for the case of $c > 2h$ which is very useful especially for comparing with the above-mentioned formulas presented in the literature.

In the second part of this article, the problem of predicting crack propagation velocity in a pin loaded DCB specimen is investigated. With the exception of the work of Kanninen (1973) and Bilek and Burns (1974), which have been analytically developed, the other solutions use numerical methods (Kanninen, 1974; Malluck and King, 1977; Gehlen et al., 1979; Popelar and Gehlen, 1979). Kanninen (1973) studied the problem of unstable crack propagation in a DCB specimen with fixed ends, considering a quasi-static crack propagation. The results when compared with experiments, showed a twofold inadequacy. First, the predicted crack speeds were greater than real speeds. Second, the results did not predict the constant speed of crack propagation.

Here, the problem is solved supposing a quasi-static crack propagation. This hypothesis, according to Baker (1972), is acceptable if the crack propagation velocity is less than $0.3C_0$, where C_0 is the longitudinal wave velocity propagation.

2. Formulation of the problem

The governing differential equation for a Timoshenko beam deflection $w(x)$ is

$$EI\frac{\partial^4 w}{\partial x^4}+m\frac{\partial^2 w}{\partial t^2}-\left(J+\frac{EIm}{\kappa AG}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2}+\frac{Jm}{\kappa AG}\frac{\partial^4 w}{\partial t^4}=p(x,t)+\frac{J}{\kappa AG}\frac{\partial^2 p}{\partial t^2}-\frac{EI}{\kappa AG}\frac{\partial^2 p}{\partial x^2} \tag{9}$$

where EI is the flexural rigidity of the beam; J and m, the rotary inertia and mass of the beam per unit length; A, the cross-sectional area; G, the shear modulus and $p(x,t)$, the applied load on the beam.

In the static case, the above equation reduces to:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}=\frac{p}{EI}-\frac{1}{\kappa AG}\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} \tag{10}$$

where $p(x)$ is the distributed load applied to the beam and κ is shear deflection coefficient of the beam and for a rectangular cross-section, according to Cowper (1966), is given by

$$\kappa=\frac{10(1+v)}{12+11v} \tag{11}$$

If the origin of coordinate is taken at the crack tip, then we have

$$p(x)=-kw(x)H(x) \tag{12}$$

where $H(x)$ is the so-called Heaviside function defined by

$$H(x)=\begin{cases}1; & x\geqslant 0\\ 0; & x<0\end{cases} \tag{13}$$

The mathematical description of the model shown in Fig. 2 can be found by substituting $p(x)$ into (10). Thus, the governing differential equation for the beam deflection $w(x)$ is

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+H(x)\left(-\frac{2(12+11v)}{5h^2}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}+\frac{24}{h^4}w\right)=0 \tag{14}$$

In order to solve Eq. (13), the two intervals $(-a,0)$ and $(0,c)$ are considered separately. In these two intervals the differential equation (14) appears as

$$\begin{cases}\dfrac{\mathrm{d}^4 w_1}{\mathrm{d}x^4}=0; & -a\leqslant x\leqslant 0\\[2ex] \dfrac{\mathrm{d}^4 w_2}{\mathrm{d}x^4}-\dfrac{2(12+11v)}{5h^2}\dfrac{\mathrm{d}^2 w_2}{\mathrm{d}x^2}+\dfrac{24}{h^4}w_2=0; & 0\leqslant x\leqslant c\end{cases} \tag{15}$$

The first equation can easily be solved to result in

$$w_1(x)=r_1\frac{x^3}{6}+r_2\frac{x^2}{2}+r_3x+r_4 \tag{16}$$

To solve the second equation, the characteristic roots should be found. After a lengthy calculation, the roots are obtained as

$$\begin{cases}D_{1,2}=\xi\pm \mathrm{i}\eta\\ D_{3,4}=-\xi\pm \mathrm{i}\eta\end{cases} \tag{17}$$

where

$$\begin{cases} \xi = \dfrac{(24)^{\frac{1}{4}}}{h}\cos\left(\dfrac{1}{2}\tan^{-1}\sqrt{\dfrac{600}{(12+11v)^2}-1}\right) \\ \eta = \dfrac{(24)^{\frac{1}{4}}}{h}\sin\left(\dfrac{1}{2}\tan^{-1}\sqrt{\dfrac{600}{(12+11v)^2}-1}\right) \end{cases} \tag{18}$$

Thus, the displacement function $w_2(x)$ can be written as

$$w_2(x) = \mathrm{e}^{\xi x}(r_5' \cos\eta x + r_6' \sin\eta x) + \mathrm{e}^{-\xi x}(r_7' \cos\eta x + r_8' \sin\eta x) \tag{19}$$

or in another form

$$w_2(x) = r_5 \sin\eta x \sinh\xi x + r_6 \sin\eta x \cosh\xi x + r_7 \cos\eta x \cosh\xi x + r_8 \cos\eta x \sinh\xi x \tag{20}$$

There are eight unknown constants in (16) and (20) which can be determined using four equations of boundary conditions and also, four equations of continuity conditions at $x = 0$. The boundary conditions corresponding to a DCB specimen subjected to splitting forces P at the end of arms are

$$\begin{cases} M(-a) = 0 \\ V(-a) = P \end{cases} \tag{21}$$

$$\begin{cases} M(c) = 0 \\ V(c) = 0 \end{cases} \tag{22}$$

The continuity conditions at $x = 0$ can be written as

$$\begin{cases} w_1(0) = w_2(0) \\ \psi_1(0) = \psi_2(0) \\ M_1(0) = M_2(0) \\ V_1(0) = V_2(0) \end{cases} \tag{23}$$

For a Timoshenko beam in the static case, the relations between the bending moment, shear force, angle of rotation and the beam deflection can be given as

$$M(x) = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{EI}{\kappa AG}p(x) \tag{24}$$

$$V(x) = EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + \frac{EI}{\kappa AG}\frac{\mathrm{d}p(x)}{\mathrm{d}x} \tag{25}$$

$$\psi(x) = \frac{\mathrm{d}w}{\mathrm{d}x} + \frac{EI}{\kappa AG}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + \frac{EI}{(\kappa AG)^2}\frac{\mathrm{d}p(x)}{\mathrm{d}x} \tag{26}$$

The details of determining unknown constants are unimportant and also very lengthy. However, the resultant relations for the unknown constants are given in Appendix A.

3. The stress intensity factor in general case

The stress intensity factor is determined with the aid of compliance approach. If $\delta = w_1(-a)$, then

$$\delta = CP \tag{27}$$

where C is the specimen compliance. The function φ is defined as below

$$C = \frac{\delta}{P} = \frac{\varphi}{Eb} \tag{28}$$

The elastic energy of the whole specimen is twice that of the model, therefore,

$$U = P\delta = \frac{P^2}{Eb}\phi \tag{29}$$

Hence

$$\Im = \frac{1}{b}\frac{dU}{da} = \frac{P^2}{Eb^2}\frac{d\phi}{da} \tag{30}$$

From the relation $K^2 = E\Im$, we have

$$K^2 = \frac{P^2}{b^2}\frac{d\phi}{da} \tag{31}$$

Since φ is a function of both a and c, which are related together through the relation $a + c = L$, we can write

$$\frac{d\phi}{da} = \frac{\partial\phi}{\partial a} - \frac{\partial\phi}{\partial c} \tag{32}$$

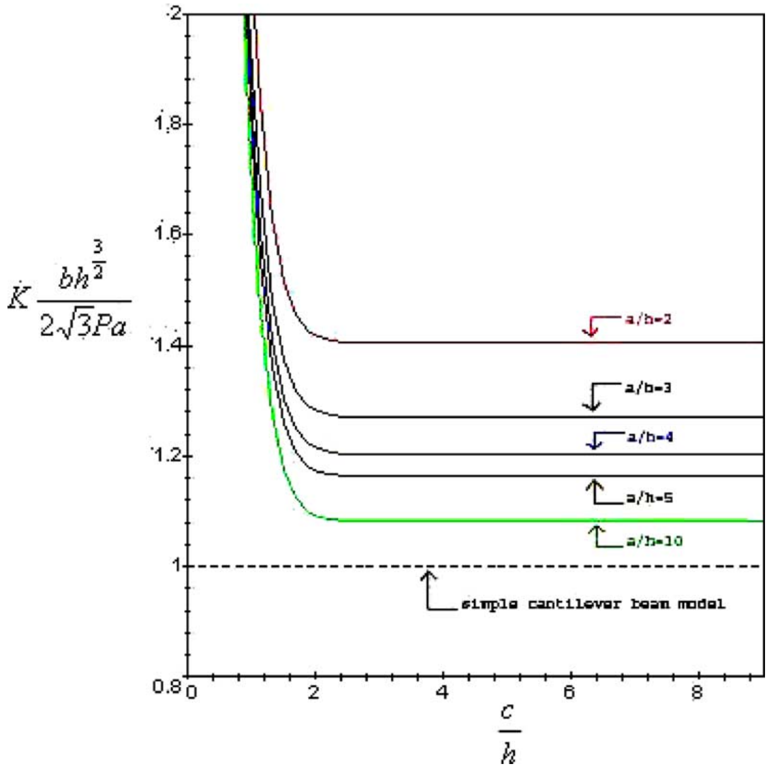

Fig. 3. Variations of the non-dimensionalized stress intensity factor as a function of $\frac{c}{h}$.

Substituting Eq. (32) into Eq. (31), yields

$$K = \frac{P}{b}\left(\frac{\partial\phi}{\partial a} - \frac{\partial\phi}{\partial c}\right)^{1/2} \tag{33}$$

Substituting for $\delta = w_1(-a)$ from Eq. (16) into Eq. (28), the function φ can be computed. Replacing φ into Eq. (33) leads to a relation for the stress intensity factor, which is not given for the sake of brevity. The resulting non-dimensionalized stress intensity factor, $K\frac{bh^{3/2}}{2\sqrt{3}Pa}$ is plotted in Fig. 3 as a function of $\frac{c}{h}$, for different values of $\frac{a}{h}$. It is observed from Fig. 3 that the stress intensity factor becomes independent of $\frac{c}{h}$ for the values $c > 2h$. However, for small values of $\frac{c}{h}$, the stress intensity factor tends to infinity. This is because the finite boundary at $x = c$ approaches the crack tip. On the other hand, it is seen from the figure that for large values of $\frac{a}{h}$, the stress intensity factor approaches to that obtained by Gilman (1959), i.e., Eq. (1), as expected.

A comparison between the present solution and Kanninen's solution (1973) is shown in Fig. 4.

The difference is due to the effect of shear deformation. It is seen that this difference decreases as $\frac{a}{h}$ increases, a result which is in agreement with the fact that the less $\frac{a}{h}$ values, the more shear deformation effects.

Fig. 5 shows the variations of non-dimensionalized stress intensity factor in the form $K\frac{b\sqrt{h}}{P}$ as a function of $\frac{a}{h}$ for different values of $\frac{c}{h}$. It is seen that the results are linear.

For the sake of comparison, the stress intensity factor is non-dimensionalized in the form $K\frac{b\sqrt{h}}{P}$ and is plotted versus $\frac{a}{h}$ for different values of $\frac{c}{h}$, together with Kanninen's results (1973) in Fig. 6.

According to Figs. 3, 5 and 6, the influence of c is very strong when $c < 2h$, but the results are quite insensitive to c when $c > 2h$. This point can lead to a simple and short relation for the case of $c > 2h$.

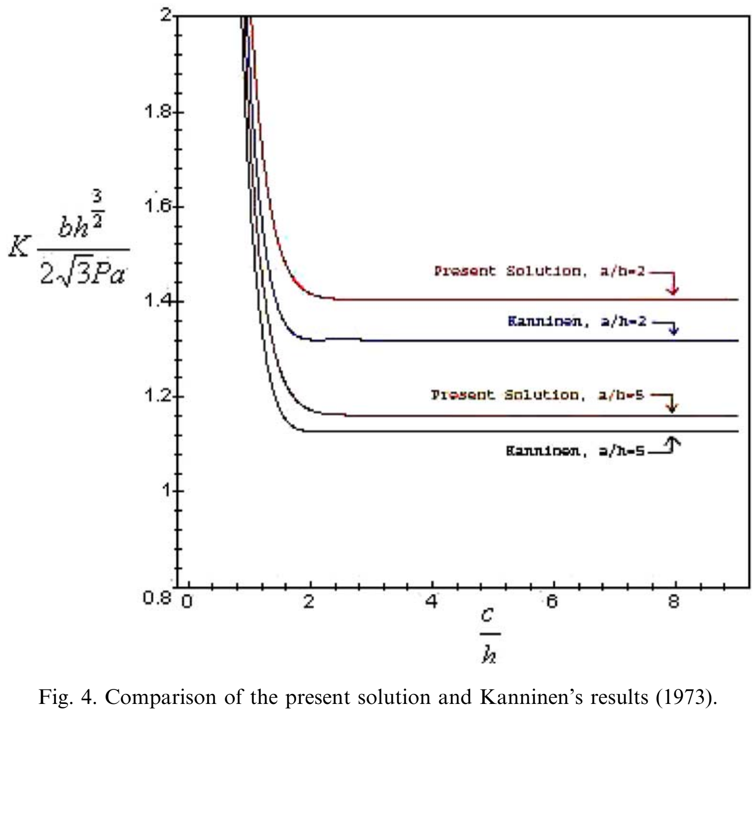

Fig. 4. Comparison of the present solution and Kanninen's results (1973).

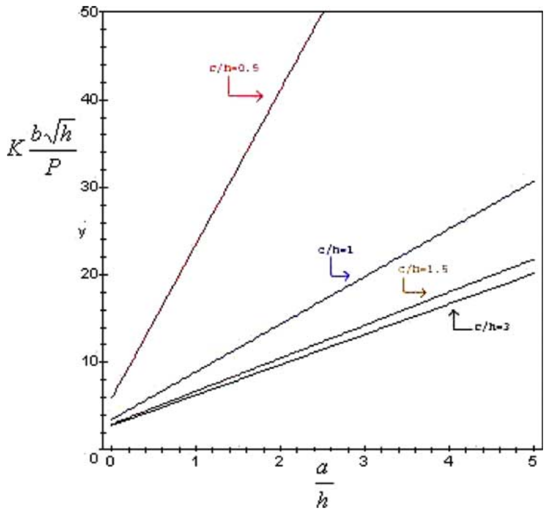

Fig. 5. Variations of the non-dimensionalized stress intensity factor as a function of $\frac{a}{h}$.

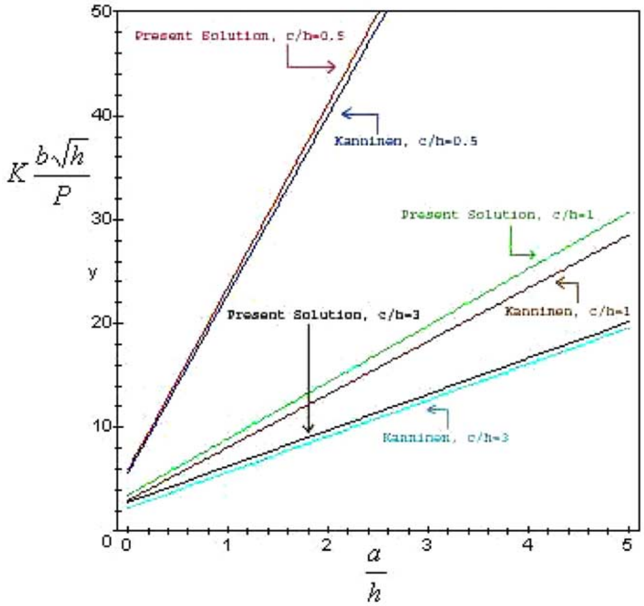

Fig. 6. Comparison of the present solution and Kanninen's results (1973).

4. Stress intensity factor for $c > 2h$

Because of insensitivity of the results to the parameter $\frac{c}{h}$ for the case $\frac{c}{h} > 2$, the beam model can be considered to extend to infinity. So, in order to guarantee that the displacement is bounded for large values of x, the constants r_5 and r_6 must be put zero in Eq. (19):

$$w_2(x) = \mathrm{e}^{-\xi x}(r_7 \cos \eta x + r_8 \sin \eta x); \quad x \geqslant 0 \tag{34}$$

The boundary and continuity conditions are the same as (21) and (23). Therefore, the displacements can be obtained by determining six unknown constants with the six above-mentioned equations to have

$$w_1(x) = \frac{12P}{Ebh^3}\left\{\frac{x^3}{6} + a\frac{x^2}{2} - \left[\frac{2a\xi(\xi^2+\eta^2) + \xi^2 + \eta^2 + \frac{2(12+11\nu)}{5h^2}}{(\xi^2+\eta^2)^2}\right]x + \left[\frac{a\left(3\xi^2 - \eta^2 - \frac{2(12+11\nu)}{5h^2}\right) + 2\xi}{(\xi^2+\eta^2)^2}\right]\right\} \tag{35}$$

$$w_2(x) = \frac{12P}{Ebh^3}\mathrm{e}^{-\xi x}\left\{\frac{a\left(\xi^2 - 3\eta^2 - \frac{2(12+11\nu)}{5h^2}\right) + 2\xi}{(\xi^2+\eta^2)^2}\cos \eta x + \frac{a\xi\left(\xi^2 - 3\eta^2 - \frac{2(12+11\nu)}{5h^2}\right) + \xi^2 - \eta^2 - \frac{2(12+11\nu)}{5h^2}}{\eta(\xi^2+\eta^2)^2}\sin \eta x\right\} \tag{36}$$

The stress intensity factor can be found similar to the previous section. First, the function φ is given by

$$\phi = \frac{12}{h^3}\left\{\frac{a^3}{3} + a\left[\frac{2a\xi(\xi^2+\eta^2) + \xi^2 + \eta^2 + \frac{2(12+11\nu)}{5h^2}}{(\xi^2+\eta^2)^2}\right] + \left[\frac{a\left(3\xi^2 - \eta^2 - \frac{2(12+11\nu)}{5h^2}\right) + 2\xi}{(\xi^2+\eta^2)^2}\right]\right\} \tag{37}$$

Substituting (37) into (33), leads to

$$K = 2\sqrt{3}\frac{Pa}{bh^{3/2}}\left(1 + \sqrt{\frac{1}{\sqrt{6}} + \frac{12+11\nu}{60}}\frac{h}{a}\right) \tag{38}$$

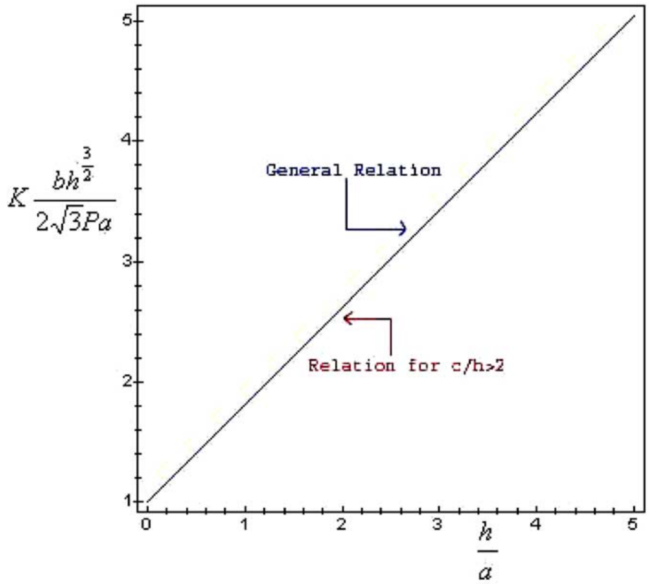

Fig. 7. Comparison between general relation and relation for $c > 2h$.

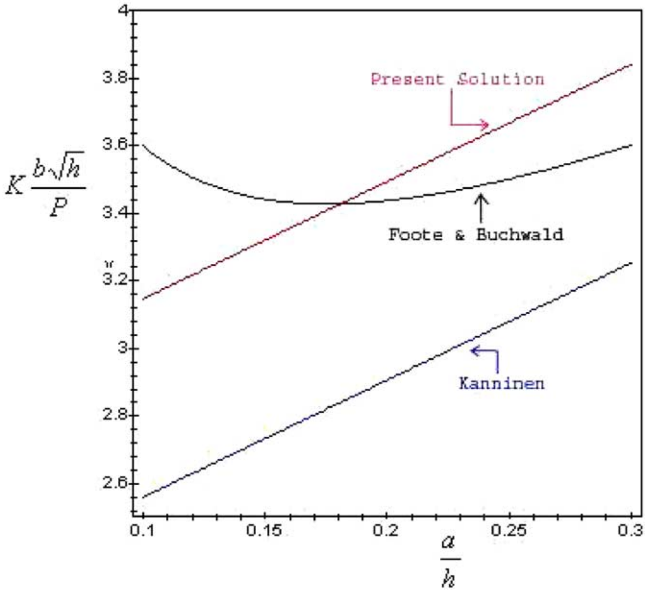

Fig. 8. Studying validity of the results for small values of $\frac{a}{h}$.

It is seen that we arrive at a closed form relation for the stress intensity factor in this case, which depends on the Poisson's ratio of the specimen. A comparison between this relation and the stress intensity factor in general case (results of the previous section) for $c = 5h$, is shown in Fig. 7. It is clear that both equations coincide completely, as expected.

In the present article, the obtained relations for the stress intensity factor are valid if the crack length is big enough to treat the model as a beam. But as the Timoshenko beam model takes the shear deformation into account, we know that the smaller the $\frac{a}{h}$ value, the more the shear deformation effects. Therefore, we expect that the range of validity of the obtained results is extended by the present solution for the stress intensity factor. It was mentioned that Foote and Buchwald's relation (1985) is valid for all crack lengths. From Fig. 8, it is apparent that for smaller cracks, the discrepancy between Kanninen's solution (1973) and Foote and Buchwald's gets higher but the present solution is valid for a greater range of $\frac{a}{h}$.

5. Dynamic crack propagation

The DCB specimen is supposed to have an initial crack length of a_0. As was mentioned in Section 1, the problem of predicting crack propagation velocity is solved considering a quasi-static crack propagation. Therefore, statical results of the previous section can be used. Substituting $v = \frac{3}{11}$ in Eqs. (35), (36) and (26), yields

$$w_1(x) = \frac{12P}{Ebh^3}\left\{\frac{x^3}{6} + \frac{ax^2}{2} - (0.8113ah + 0.4541h^2)x + (0.2041ah^2 + 0.1656h^3)\right\} \tag{39}$$

$$w_2(x) = \frac{12P}{Ebh^3}e^{-\xi x}\{(0.2041ah^2 + 0.1656h^3)\cos\eta x - (0.4163ah^2 + 0.1283h^3)\sin\eta x\} \tag{40}$$

$$\psi_1(x) = \frac{12P}{Ebh^3}\left\{\frac{1}{2}x^2 + ax - (0.8113ah + 0.2041h^2)\right\} \tag{41}$$

$$\psi_2(x) = \frac{12P}{Ebh^3}e^{-\xi x}\{-(0.8113ah + 0.2041h^2)\cos(\eta x) - (0.6284ah + 0.4163h^2)\sin(\eta x)\} \tag{42}$$

In this section, the origin of coordinate is taken at the point where the force applies. Therefore, it is needed to replace x with $x - a$. Thus, we have

$$w_1(x) = \frac{12P}{Ebh^3}\left\{\frac{1}{6}(x-a)^3 + \frac{1}{2}a(x-a)^2 - (0.8113ah + 0.4541h^2)(x-a) + (0.2041ah^2 + 0.1656h^3)\right\} \tag{43}$$

$$w_2(x) = \frac{12P}{Ebh^3}e^{-\xi(x-a)}\{(0.2041ah^2 + 0.1656h^3)\cos\eta(x-a) - (0.4163ah^2 + 0.1283h^3)\sin\eta(x-a)\} \tag{44}$$

$$\psi_1(x) = \frac{12P}{Ebh^3}\left\{\frac{1}{2}(x-a)^2 + a(x-a) - (0.8113ah + 0.2041h^2)\right\} \tag{45}$$

$$\psi_2(x) = \frac{12P}{Ebh^3}e^{-\xi(x-a)}\{-(0.811ah + 0.2041h^2)\cos(\eta(x-a)) - (0.6284ah + 0.4163h^2)\sin(\eta(x-a))\} \tag{46}$$

The energy components and the work done during crack propagation can be computed using the following relations

$$U = \int_0^\infty \left\{EI\left(\frac{\partial\psi}{\partial x}\right)^2 + \kappa AG\left(\frac{\partial w}{\partial x} - \psi\right)^2 + H(x-a)kw^2\right\}dx \tag{47}$$

$$T = \int_0^\infty \left\{\rho A\left(\frac{\partial w}{\partial t}\right)^2 + \rho I\left(\frac{\partial\psi}{\partial t}\right)^2\right\}dx \tag{48}$$

$$W = 2Pw_1(0) \tag{49}$$

The strain energy is then obtained by substituting Eqs. (43)–(46) into (47). After some mathematical manipulations, we have

$$U = \frac{P^2}{Eb}\left\{1.9873 + 7.899\left(\frac{a}{h}\right) + 9.7359\left(\frac{a}{h}\right)^2 + 4\left(\frac{a}{h}\right)^3\right\} \tag{50}$$

Considering the following relations

$$\frac{\partial w_i}{\partial t} = \frac{\partial w_i}{\partial a}\frac{\partial a}{\partial t} = V\frac{\partial w_i}{\partial a}; \quad i = 1,2 \tag{51}$$

$$\frac{\partial \psi_i}{\partial t} = \frac{\partial \psi_i}{\partial a}\frac{\partial a}{\partial t} = V\frac{\partial \psi_i}{\partial a}; \quad i = 1,2 \tag{52}$$

where V is the crack propagation velocity, the kinetic energy becomes

$$T = 12\frac{P^2}{Eb}\left(\frac{V}{C_0}\right)^2\left\{1.2067 + 8.8325\left(\frac{a}{h}\right) + 8.8325\left(\frac{a}{h}\right)^2 + 22.6818\left(\frac{a}{h}\right)^3 + 16.2265\left(\frac{a}{h}\right)^4 + 4\left(\frac{a}{h}\right)^5\right\} \tag{53}$$

And finally, the work done during crack propagation can be obtained by substituting Eq. (43) into (49):

$$W = \frac{P^2}{Eb}\left\{3.9747 + 15.7979\left(\frac{a}{h}\right) + 19.4718\left(\frac{a}{h}\right)^2 + 8\left(\frac{a}{h}\right)^3\right\} \tag{54}$$

A statement of energy conservation is

$$[W(a) - W(a_0)] - [U(a) - U(a_0)] - T(a) = b\int_{a_0}^{a} R(a)\,\mathrm{d}a \tag{55}$$

where R is the energy absorption per unit area during crack extension. In the present work, it will be considered that R is constant during crack propagation. Thus,

$$R = \begin{cases} R_s; & V = 0 \\ R_d; & V \neq 0 \end{cases} \tag{56}$$

where R_s and R_d are the crack resistances against the extension in the static and dynamic cases, respectively. Obviously, $R_d < R_s$ for crack propagation to be possible.

Substituting Eq. (56) into (55), gives

$$T(a) = [W(a) - W(a_0)] - [U(a) - U(a_0)] - bR_d(a - a_0) \tag{57}$$

At the threshold of crack extension R_s must be equal to the energy release rate. Therefore, using Eq. (30), R_s can be written as

$$R_s = \frac{P^2}{Eb^2}\frac{\mathrm{d}\phi}{\mathrm{d}a}\bigg|_{a_0} = \frac{12P^2}{Eb^2}\left(\frac{a_0}{h} + 0.8113\right)^2 \tag{58}$$

Therefore, the last term of Eq. (57) can be written with the aid of Eq. (58) as

$$\begin{aligned} bR_d(a - a_0) &= b(a - a_0)\frac{R_d}{R_s}\frac{12P^2}{Eb^2}\left(\frac{a_0}{h} + 0.8113\right)^2 \\ &= \frac{P^2}{Eb}\frac{a - a_0}{h}\left(7.899 + 19.4712\left(\frac{a_0}{h}\right) + 12\left(\frac{a_0}{h}\right)^2\right)\frac{R_d}{R_s} \end{aligned} \tag{59}$$

Substituting Eqs. (50), (54) and (59) into (57), leads to

$$\begin{aligned} T(a) = \frac{P^2}{Eb}\Bigg\{&7.899\left(\frac{a - a_0}{h}\right) + 9.7359\left(\frac{a^2 - a_0^2}{h^2}\right) + 4\left(\frac{a^3 - a_0^3}{h^3}\right) \\ &- \left[\left(\frac{a - a_0}{h}\right)\left(7.899 + 19.4712\left(\frac{a_0}{h}\right) + 12\left(\frac{a_0}{h}\right)^2\right)\frac{R_d}{R_s}\right]\Bigg\} \end{aligned} \tag{60}$$

Equating $T(a)$ from Eqs. (53) and (60), the crack propagation velocity can be obtained

$$\frac{V}{C_0} = \frac{A}{B} \tag{61}$$

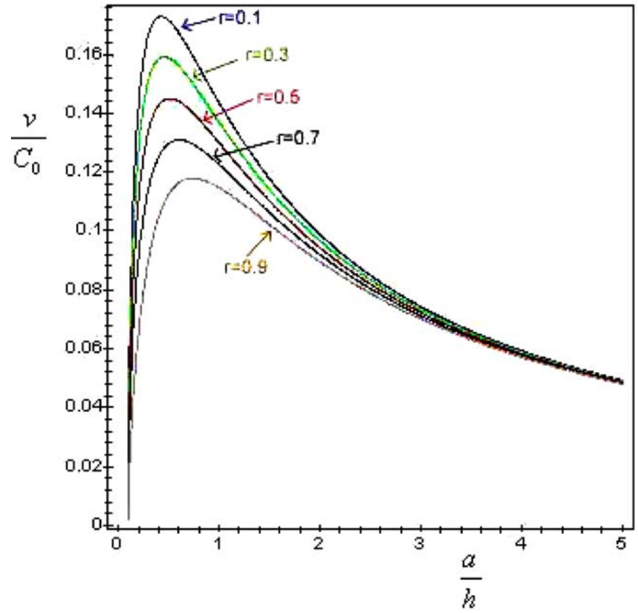

Fig. 9. Variations of the crack propagation velocity versus $\frac{a}{h}$ for different r-values.

where

$$A = 7.899\left(\frac{a-a_0}{h}\right) + 9.7359\left(\frac{a^2-a_0^2}{h^2}\right) + 4\left(\frac{a^3-a_0^3}{h^3}\right) - \left\{\left(\frac{a-a_0}{h}\right)\left[7.899 + 19.4712\left(\frac{a_0}{h}\right) + 12\left(\frac{a_0}{h}\right)^2\frac{R_d}{R_s}\right]\right\} \quad (62)$$

$$B = 12\left\{1.2067 + 8.8325\left(\frac{a}{h}\right) + 22.6818\left(\frac{a}{h}\right)^2 + 27.3299\left(\frac{a}{h}\right)^3 + 16.2265\left(\frac{a}{h}\right)^4 + 4\left(\frac{a}{h}\right)^5\right\} \quad (63)$$

Fig. 9 shows the variations of the normalized velocity, $\frac{V}{C_0}$, versus $\frac{a}{h}$ for a specimen with initial crack length of $a_0 = 0.1h$. It should be noted that r in the figure is the ratio of the crack resistances in dynamic and static crack states, i.e., $r = \frac{R_d}{R_s}$. It is observed that the crack velocity is a descending function as was predicted by Malluck and King (1977). Also, the peak values of the crack propagation velocities are less than $0.18C_0$, which is in agreement with the pre-assumption that in the $V < 0.3C_0$ range, the problem may be treated as a quasi-static one.

Appendix A

The values of unknown constants of Eqs. (16) and (20) for $v = \frac{3}{11}$ are as below

$$r_1 = \frac{12P}{Ebh^3}$$

$$r_2 = \frac{12Pa}{Ebh^3}$$

$$r_3 = \frac{P}{Eb}\left\{\frac{\sqrt{6}}{h} - \frac{2}{h}\sqrt{\sqrt{6}+\frac{3}{2}}\left(\frac{A_{13}B_{12}-A_{12}B_{13}}{A_{13}B_{24}-A_{24}B_{13}}\right)\right\}$$

$$r_4 = \frac{P}{Eb}\left(\frac{A_{24}B_{12}-A_{12}B_{24}}{A_{13}B_{24}-A_{24}B_{13}}\right)$$

$$r_5 = \frac{P}{Eb}\left\{\frac{4\sqrt{15}}{5}\frac{a}{h} + \frac{\sqrt{15}}{5}\left(\frac{A_{24}B_{12}-A_{12}B_{24}}{A_{13}B_{24}-A_{24}B_{13}}\right)\right\}$$

$$r_6 = \frac{p}{Eb}\left\{\sqrt{\frac{6}{\sqrt{6}-\frac{3}{2}}} - \sqrt{\frac{\sqrt{6}+\frac{3}{2}}{\sqrt{6}-\frac{3}{2}}}\left(\frac{A_{13}B_{12}-A_{12}B_{13}}{A_{13}B_{24}-A_{24}B_{13}}\right)\right\}$$

$$r_7 = \frac{P}{Eb}\left(\frac{A_{24}B_{12}-A_{12}B_{24}}{A_{13}B_{24}-A_{24}B_{13}}\right)$$

$$r_8 = -\frac{P}{Eb}\left(\frac{A_{13}B_{12}-A_{12}B_{13}}{A_{13}B_{24}-A_{24}B_{13}}\right)$$

where A_{12}, A_{13}, A_{24}, B_{12}, B_{24} and B_{13} are

$$A_{12} = \frac{4\sqrt{15}}{5}\frac{a}{h}A_1 + \sqrt{\frac{6}{\sqrt{6}-\frac{3}{2}}}A_2$$

$$B_{12} = \frac{4\sqrt{15}}{5}\frac{a}{h}B_1 + \sqrt{\frac{6}{\sqrt{6}-\frac{3}{2}}}B_2$$

$$A_{13} = A_3 + \frac{\sqrt{15}}{5}A_1$$

$$B_{13} = B_3 + \frac{\sqrt{15}}{5}B_1$$

$$A_{24} = A_4 + \sqrt{\frac{\sqrt{6}+\frac{3}{2}}{\sqrt{6}-\frac{3}{2}}}A_2$$

$$B_{24} = B_4 + \sqrt{\frac{\sqrt{6}+\frac{3}{2}}{\sqrt{6}-\frac{3}{2}}}B_2$$

Also A_1, A_2, A_3, B_1, A_4, B_2, B_3 are

$$A_1 = -3\sin\eta c\,\sinh\,\xi c + \sqrt{15}\cos\eta c\,\cosh\,\xi c$$

$$A_2 = -3\sin\eta c\,\cosh\,\xi c + \sqrt{15}\cos\eta c\,\sinh\,\xi c$$

$$A_3 = -3\cos\eta c\,\cosh\,\xi c - \sqrt{15}\sin\eta c\,\sinh\,\xi c$$

$$A_4 = -3\cos\eta c \sinh\xi c - \sqrt{15}\sin\eta c \cosh\xi c$$

$$B_1 = -\sqrt{\sqrt{6}+\frac{3}{2}}\sin\eta c \cosh\xi c + \sqrt{\sqrt{6}-\frac{3}{2}}\cos\eta c \sinh\xi c$$

$$B_2 = -\sqrt{\sqrt{6}+\frac{3}{2}}\sin\eta c \sinh\xi c + \sqrt{\sqrt{6}-\frac{3}{2}}\cos\eta c \cosh\xi c$$

$$B_3 = -\sqrt{\sqrt{6}+\frac{3}{2}}\cos\eta c \sinh\xi c - \sqrt{\sqrt{6}-\frac{3}{2}}\sin\eta c \cosh\xi c$$

$$B_4 = -\sqrt{\sqrt{6}+\frac{3}{2}}\cos\eta c \cosh\xi c - \sqrt{\sqrt{6}-\frac{3}{2}}\sin\eta c \sinh\xi c$$

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